X-Splines

Flexible and user-friendly

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August 24, 2018

Outline

Splines in general

X-splines

X-tended X-splines

General X-splines

Outline

What is a spline?

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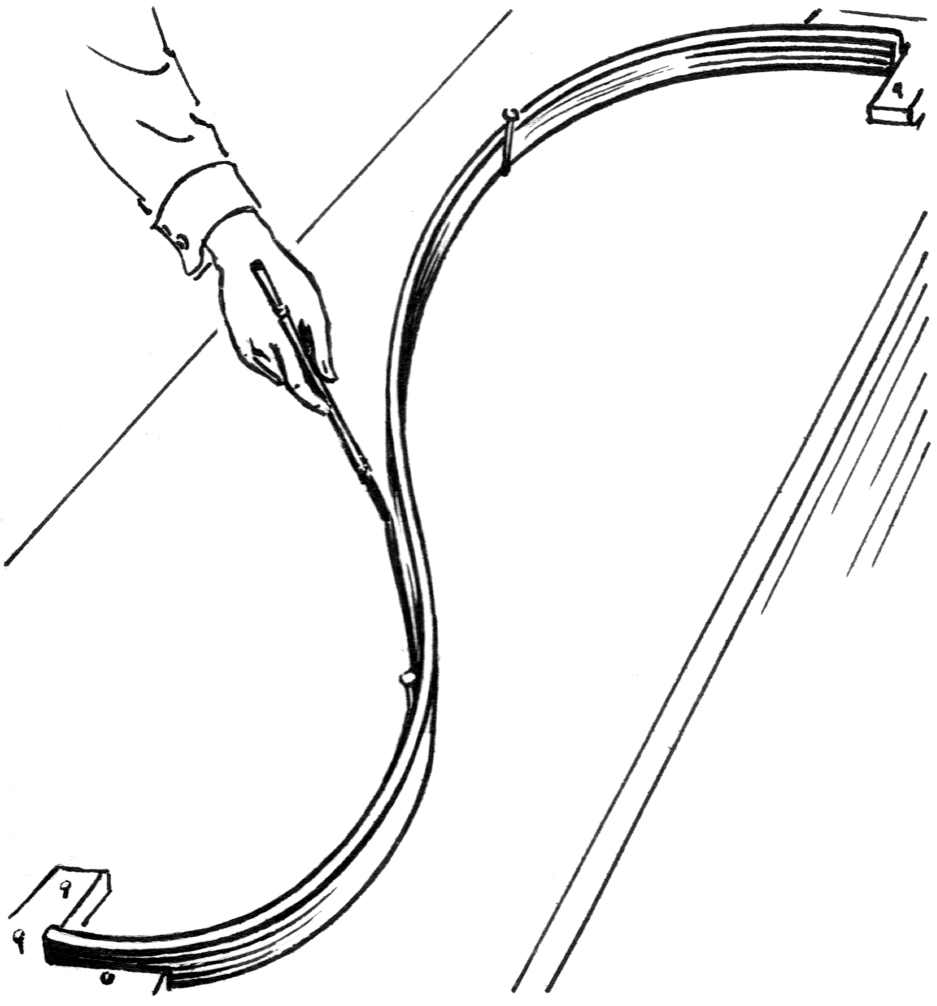
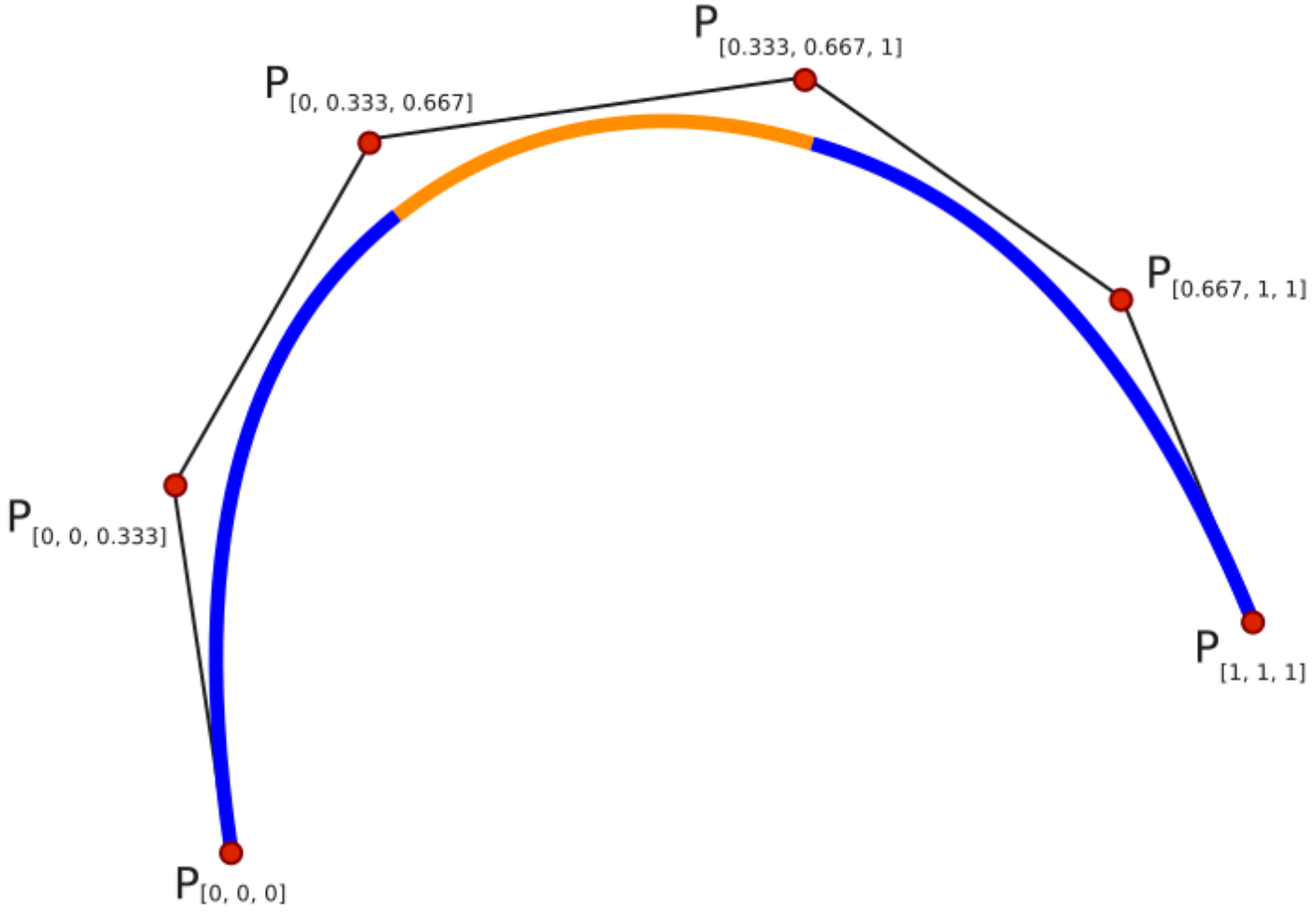


Historically: a mechanical device for drawing curves

What is a spline?

You probably know them from computer graphics or CAD.



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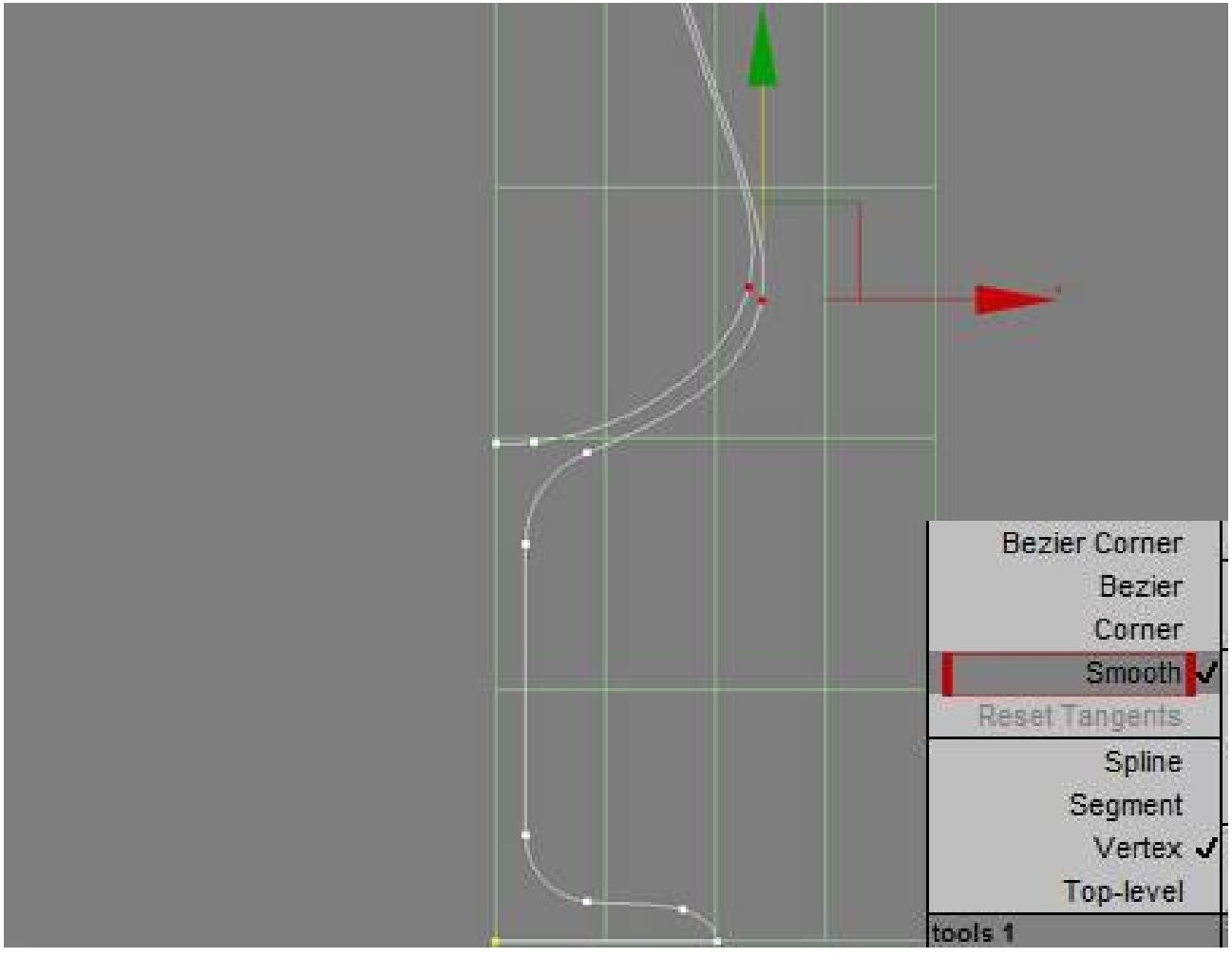
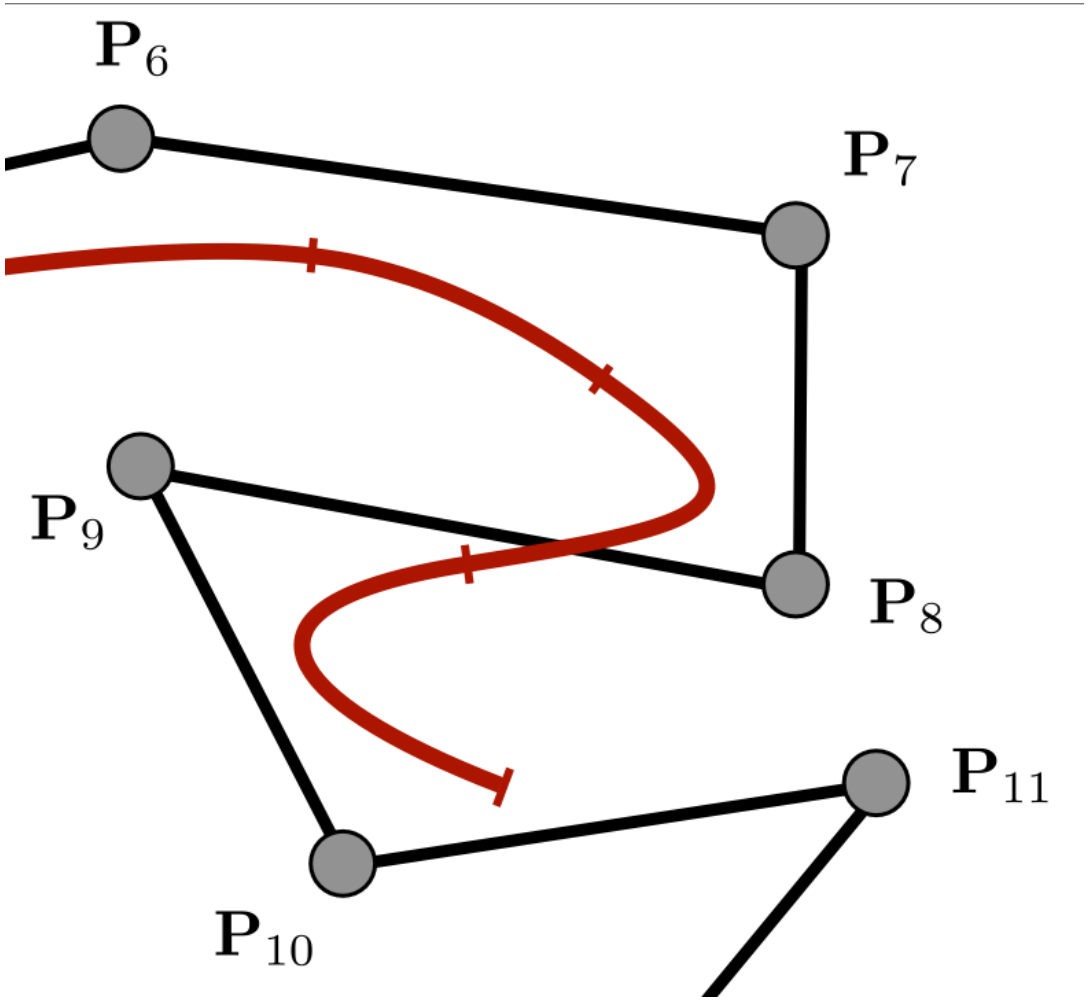


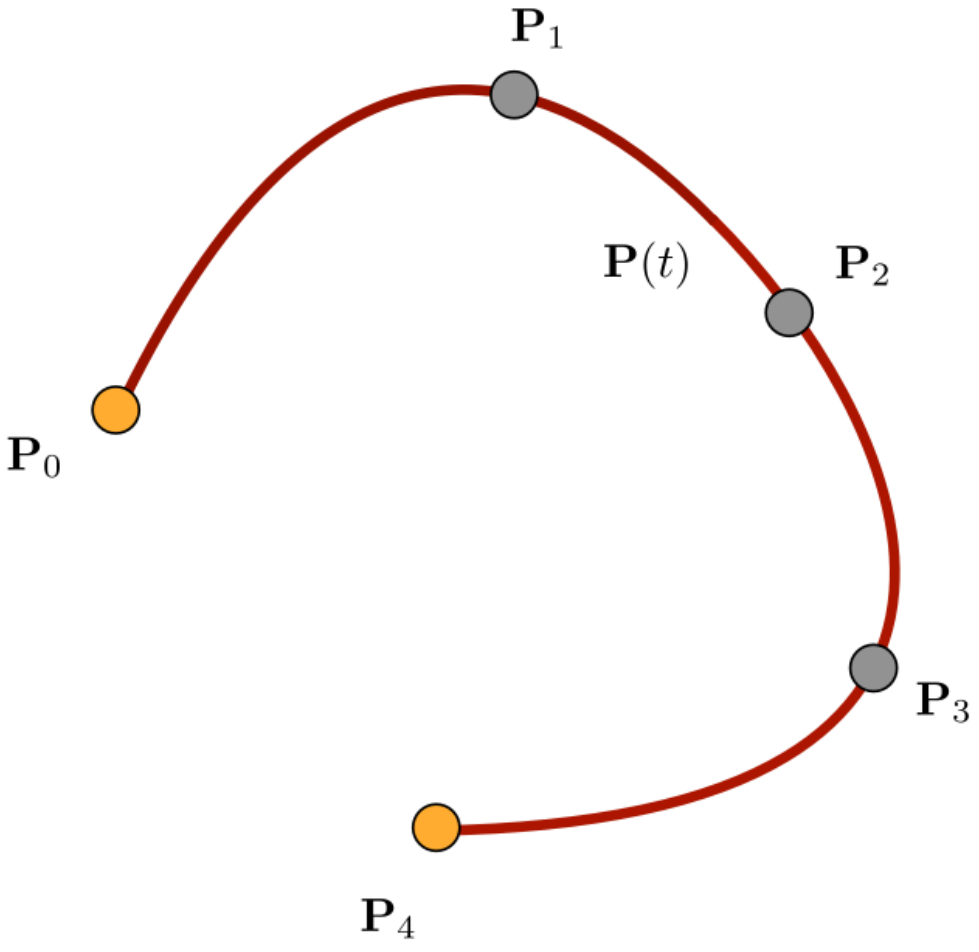
Guiding example: b-splines

These splines *approximate*.



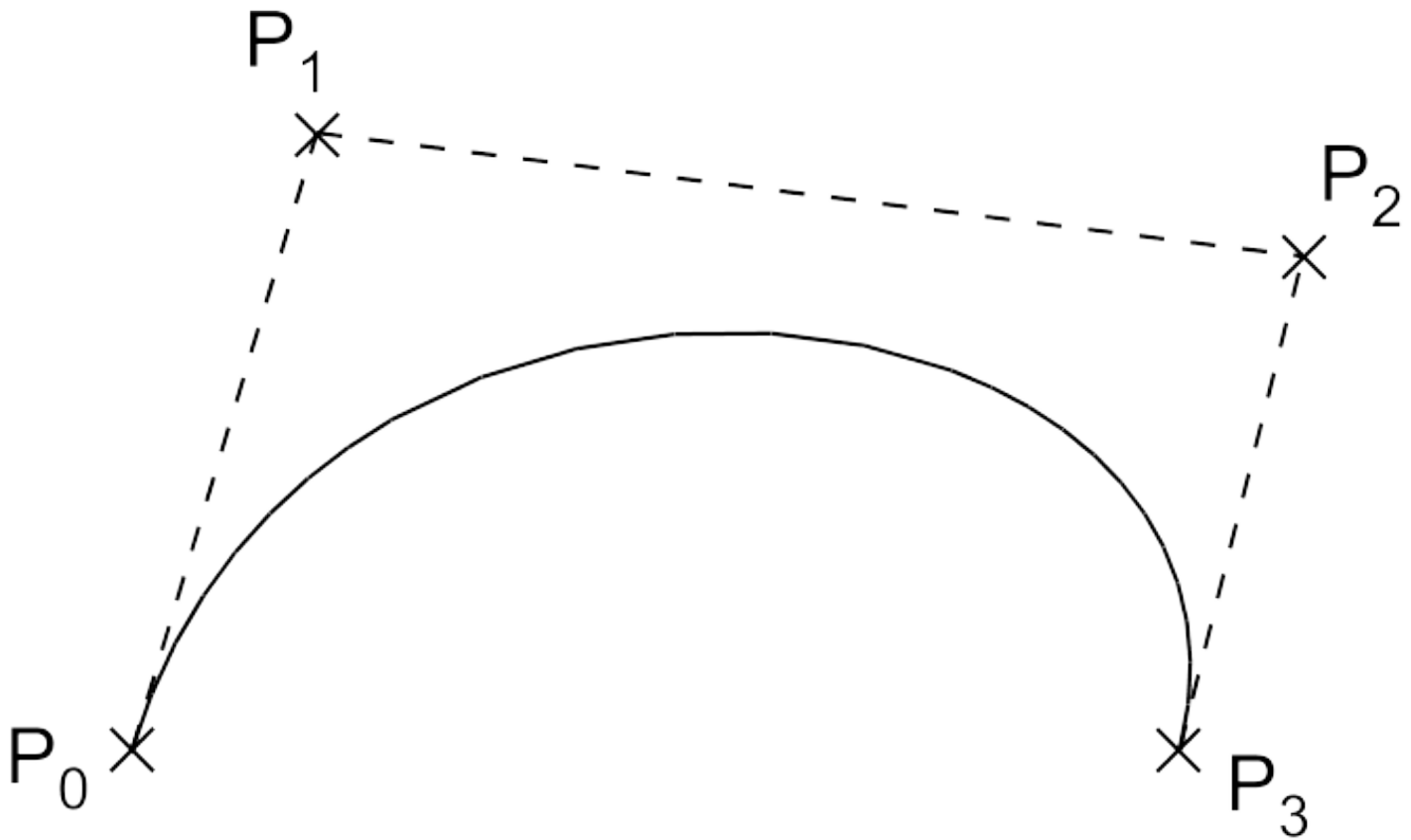
Guiding example: Catmull-Rom splines

These splines *interpolate*.



Another famous example: Bézier curves

Closely related to B-splines



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For each *node* in 2d (or 3d) space there is a corresponding *blending function*:

$$
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(x_0, y_0) = P_0 &\longleftrightarrow F_0(t) \\
(x_1, y_1) = P_1 &\longleftrightarrow F_1(t) \\
&\vdots \\
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and we form a *linear combination*

$$\begin{aligned}C(t) &= \sum_{k=0}^{n} F_k(t) P_k\\ &= F_0(t)P_0 + F_1(t)P_1 + \cdots + F_n(t)P_n.\end{aligned}$$

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The parameter t runs continuously from 0 to n.

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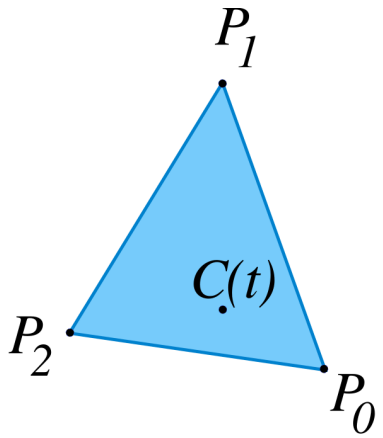
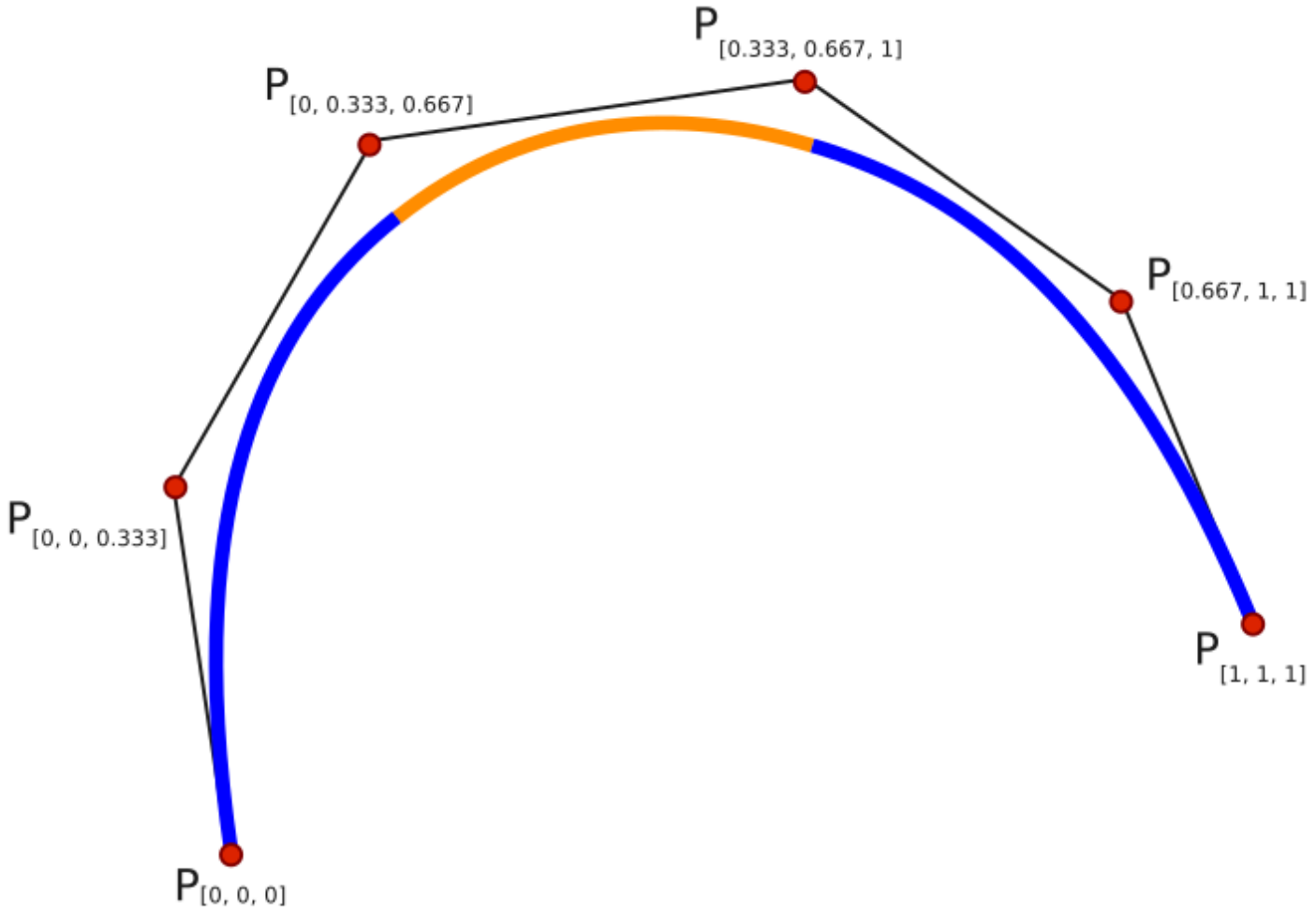
is a *convex combination* of the spline nodes.



Convexity

Remember?



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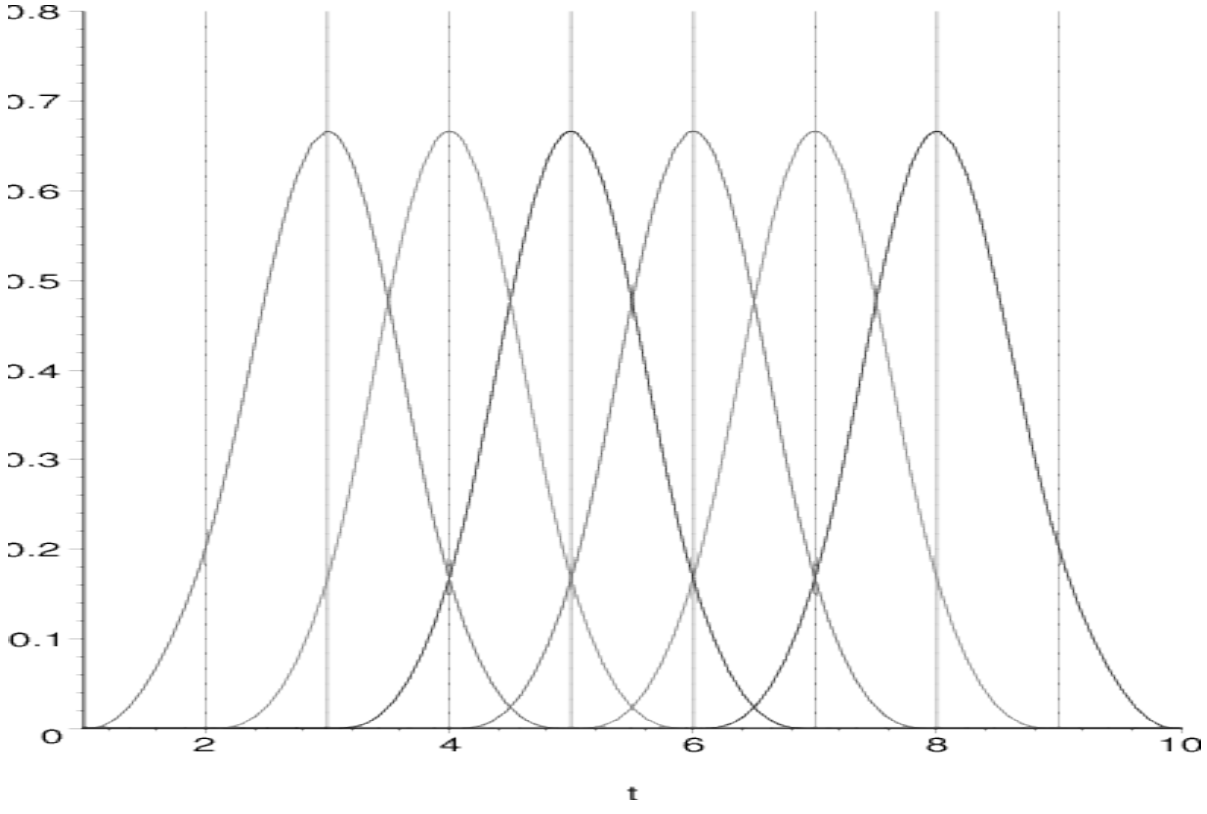

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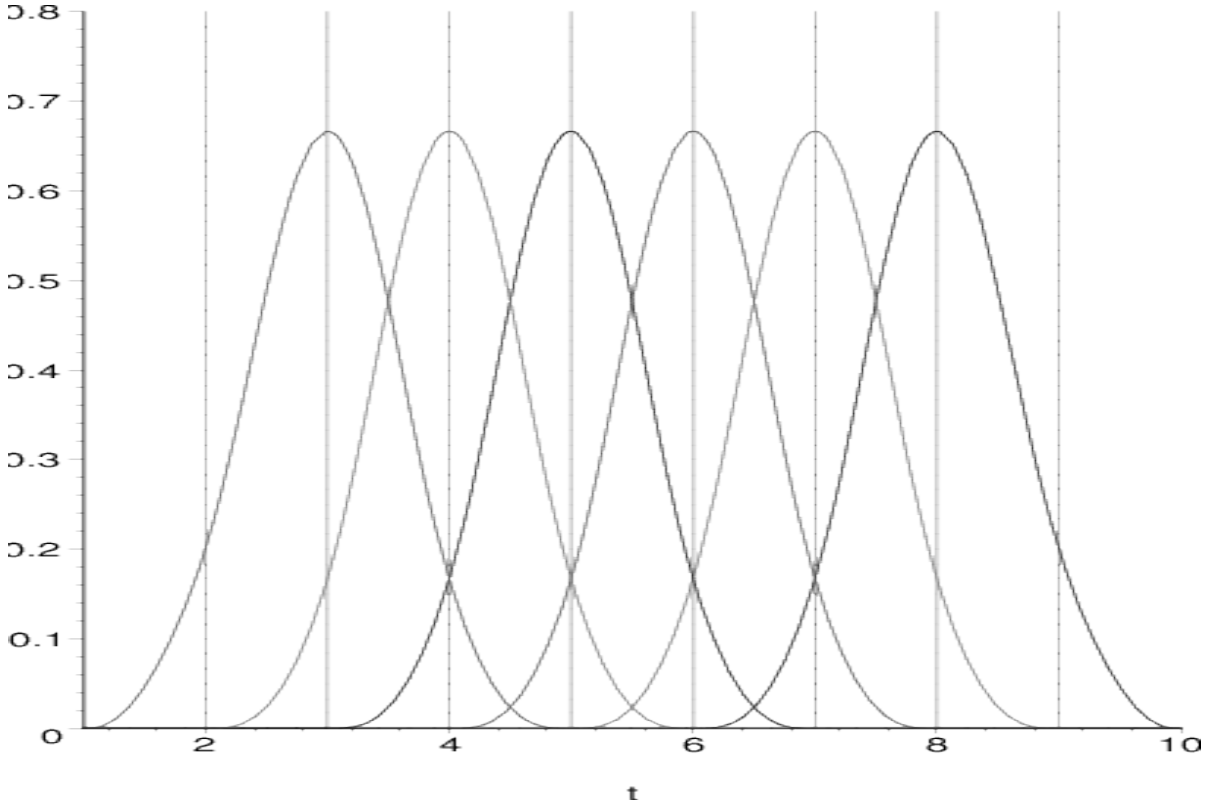

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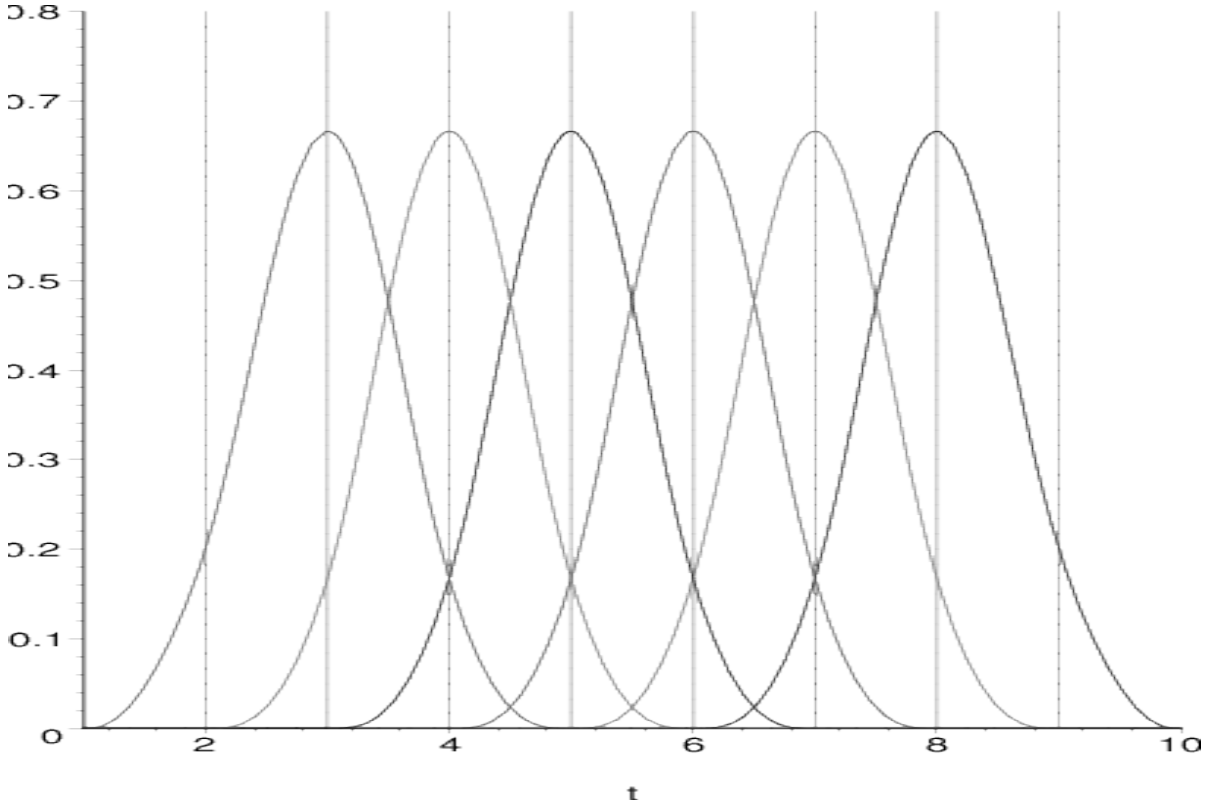

- Bump-shaped (smoothness)
- Always go in opposite direction when they cross (steadiness)

Local influence

Each blending function $F_k(t)$ must not contribute to nodes P_k that are too far away.

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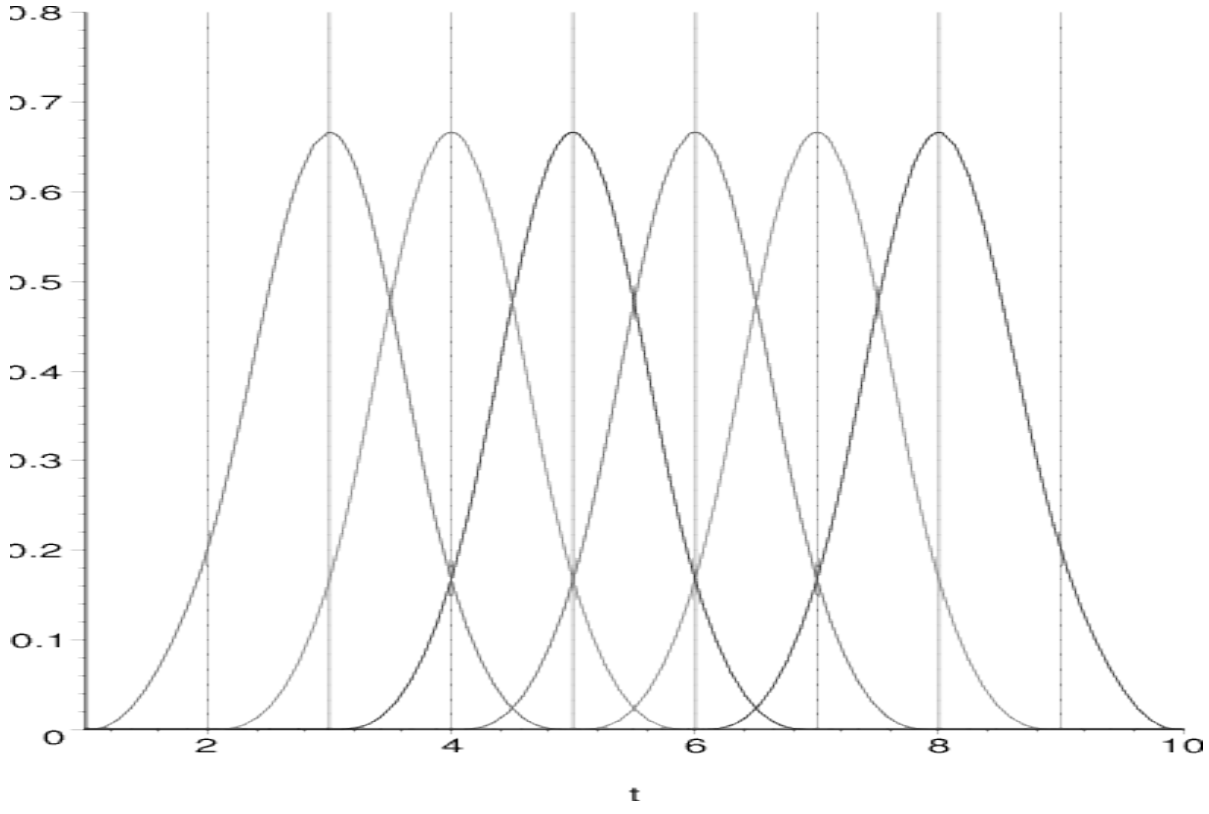
Each blending function $F_k(t)$ must not contribute to nodes P_k that are too far away.
Therefore, the blending functions are zero outside of a small zone (in technical jargon: *compact support*).



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Additional requirements

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- smoothness
- steadiness
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We want in addition

- flexibility of shape
- intuitive shape parameter
- smooth transition between approximation and interpolation

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Will begin with polynomial blending functions $F_k(t)$, e.g. one such function for Bézier curves

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$$f_k(t) = 3t^2(1-t)$$

To get convexity, (partition of unity) will normalise each blending function, i.e.

$$F_k(t) := \frac{f_k(t)}{\sum_{k=0}^{n} f_k(t)}$$

giving a *rational function* (a fraction of polynomials).

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The functions will be defined piecewise on equally-spaced intervals of the parameter space, i.e.

$$f_k(t) := \begin{cases} f\left(\frac{t-k}{2}\right) & \text{if } t \in [k-2, k] \\ f\left(\frac{k-t}{2}\right) & \text{if } t \in [k, k+2] \\ 0 & \text{otherwise} \end{cases}$$

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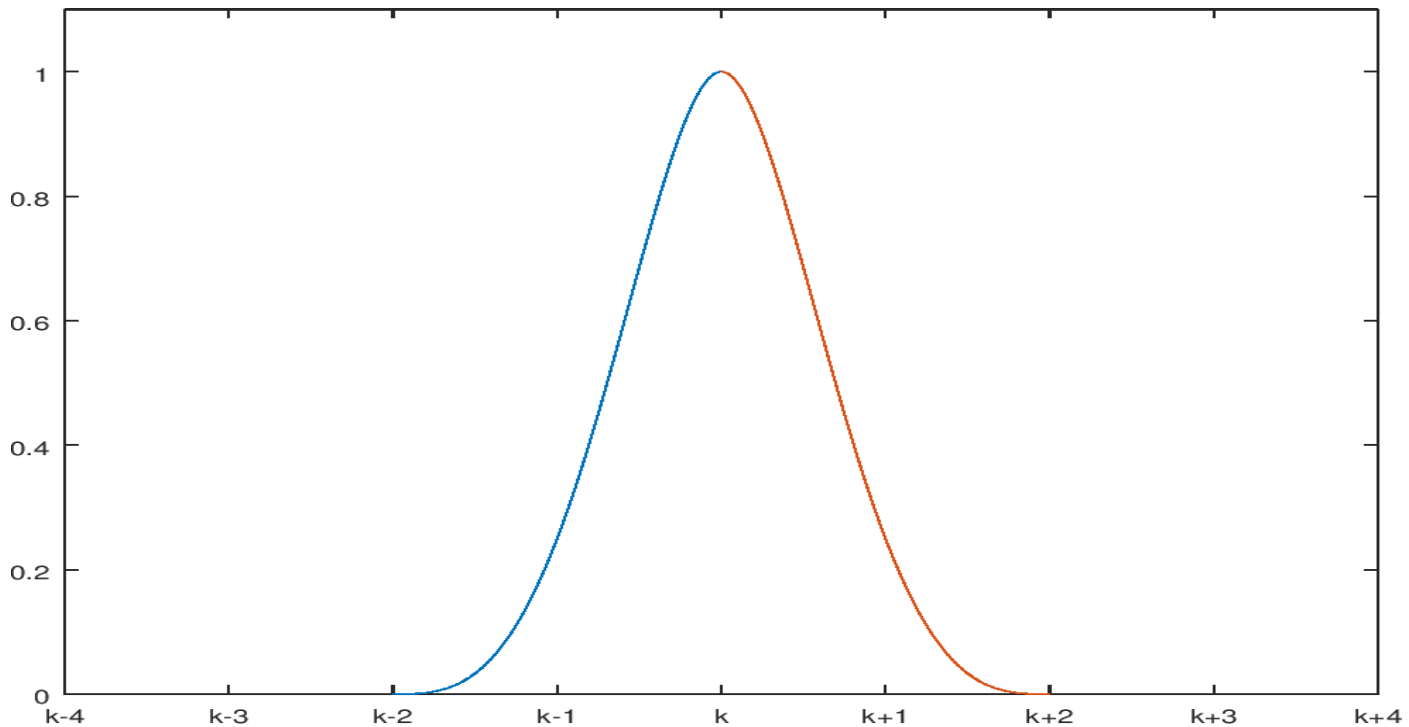

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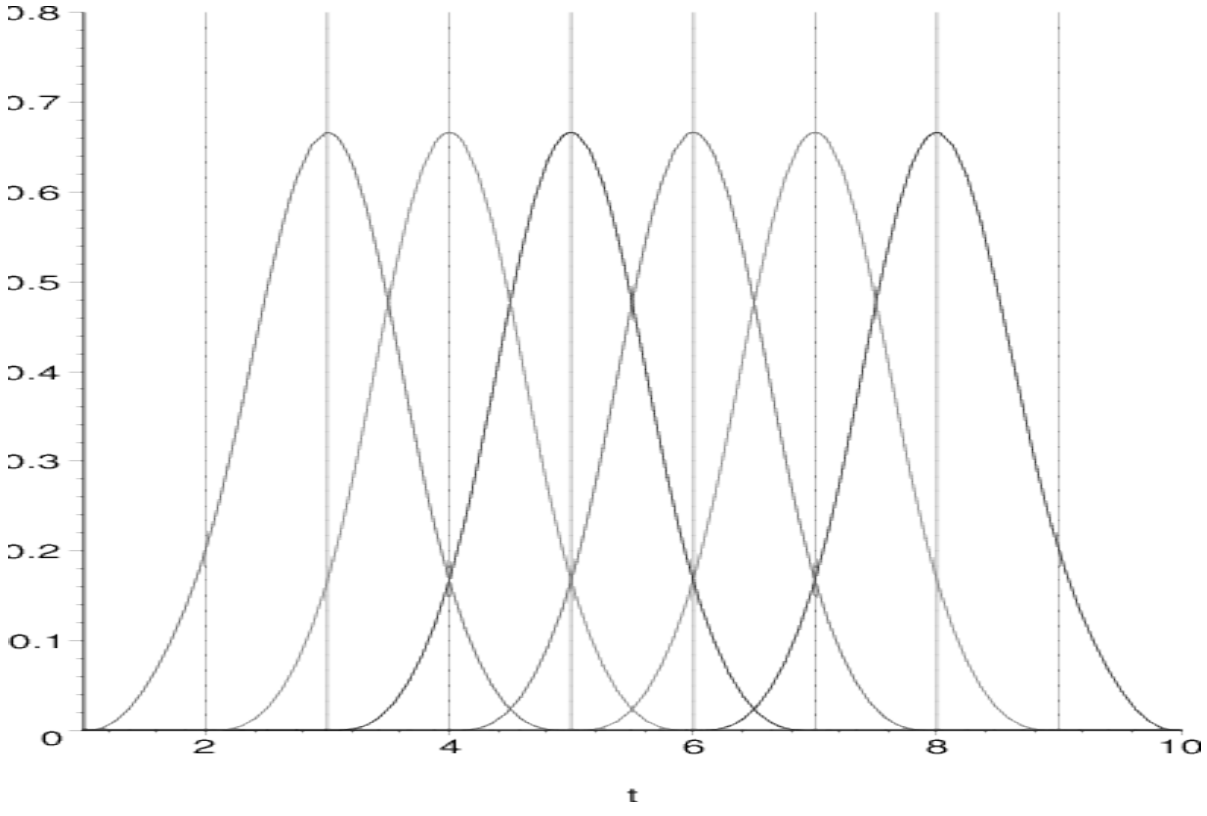
Recall: blending functions



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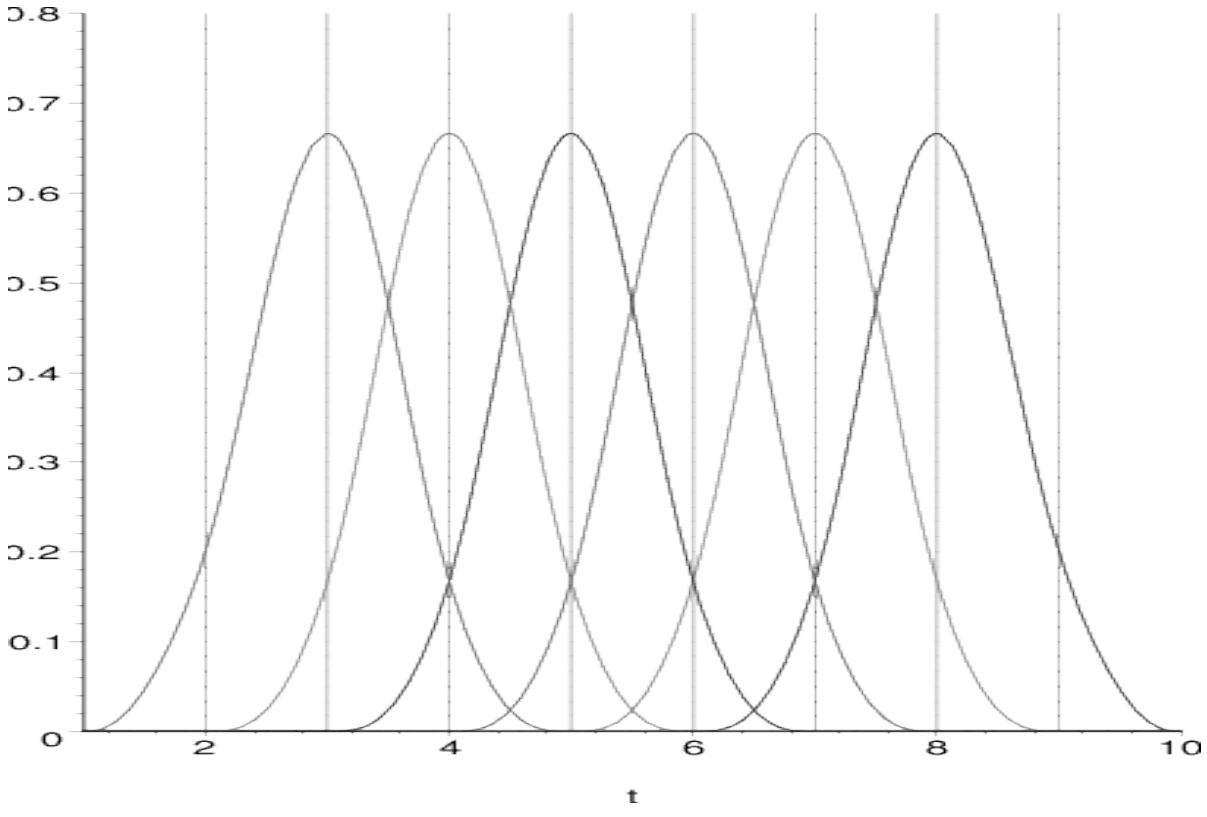

Neighbours of $F_k(t)$ cross at k, hence *cross-splines* or x-splines.

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- First derivatives (tangent lines) must be equal
- Second derivatives (curvatures) must be equal

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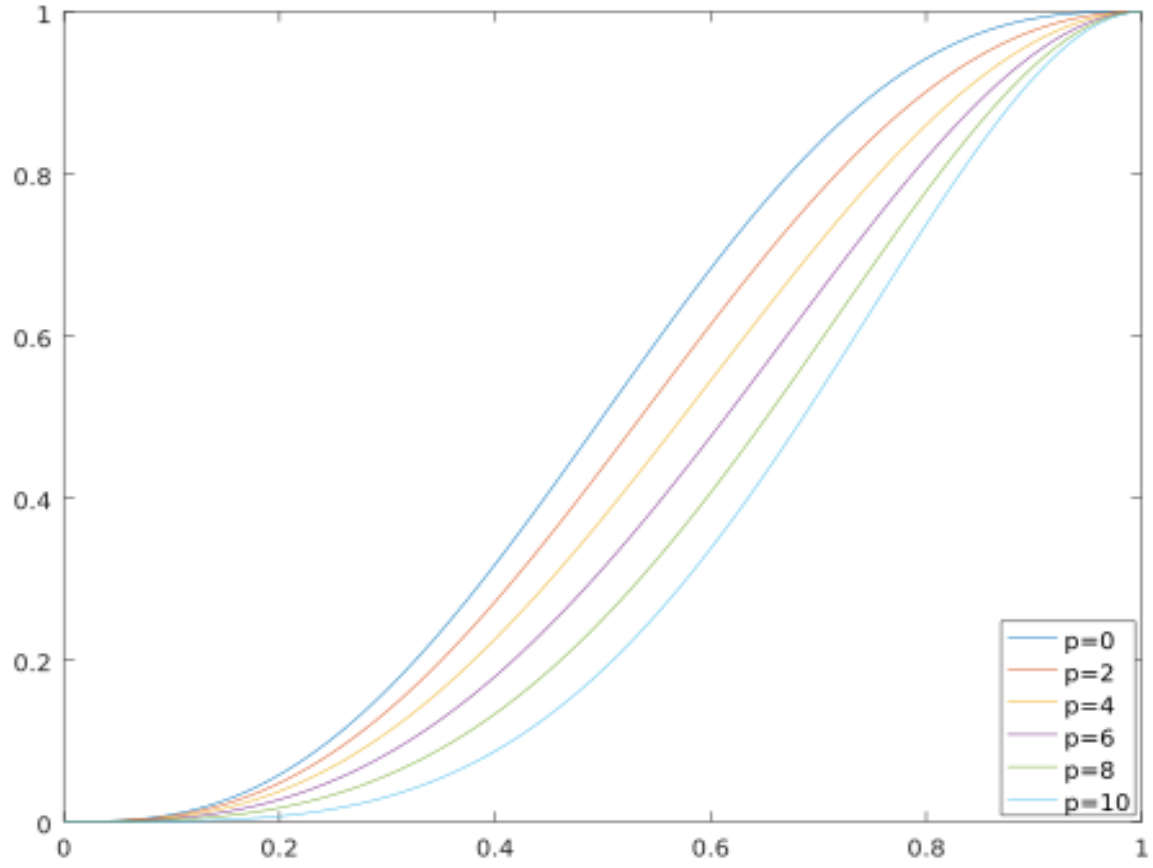
Six equations, so degree 5 polynomial. Solve to get

$$f(u)=(6-p)u^5+(2p-15)u^4+(10-p)u^3$$

The basic ingredient

As free parameter *p* varies from 0 to 10, we have some flexibility in shape:



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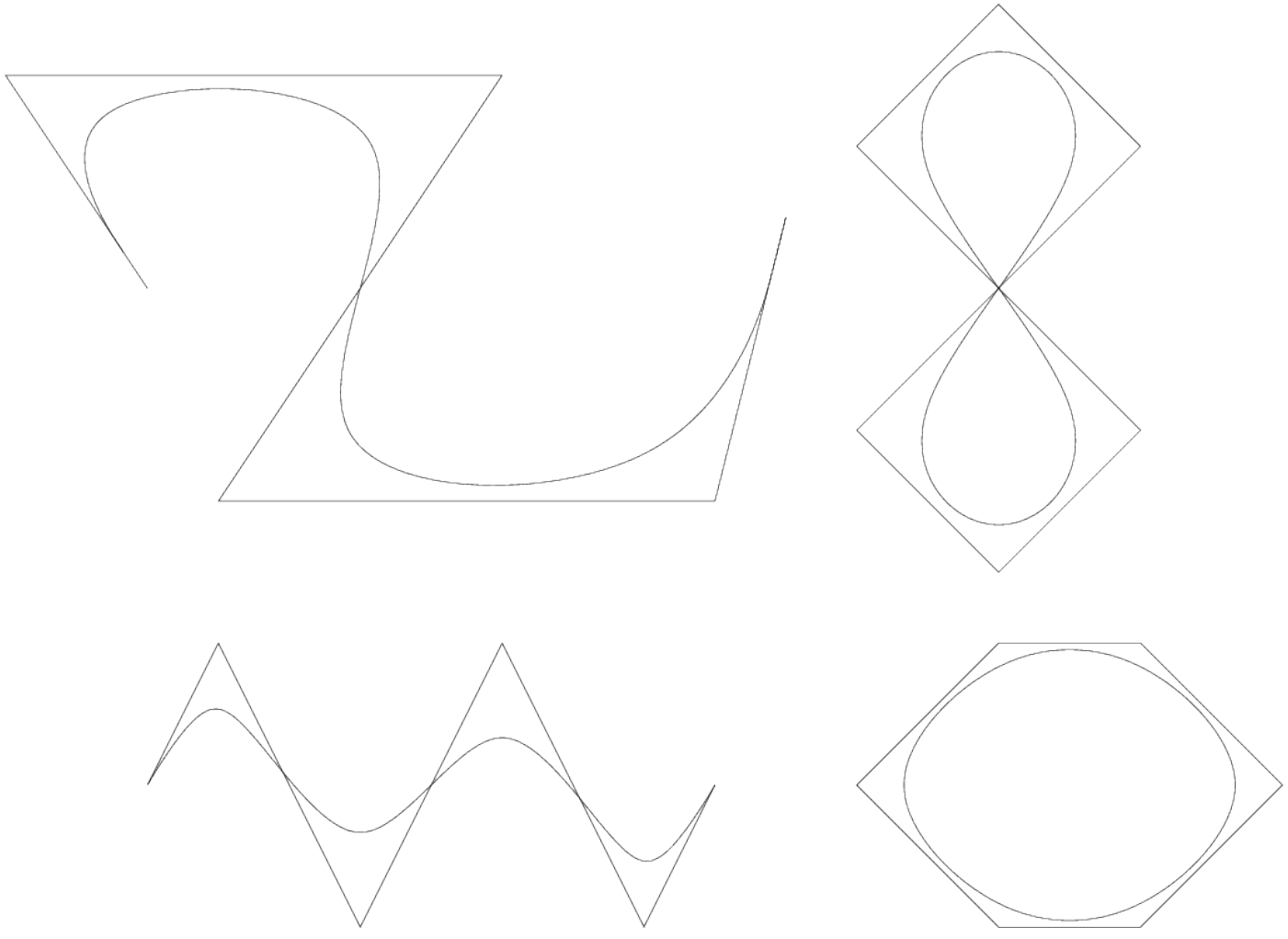
Observe that each $F_k(t)$ has its own p_k^{Left} and p_k^{Right}parameters too!

The basic x-spline

Setting all $p = 8$, we get a very good approximation of B-splines.



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Pushing the blending function upwards

The parameters p_k^{Left} and p_k^{Right} give us some freedom to modify the spline more.

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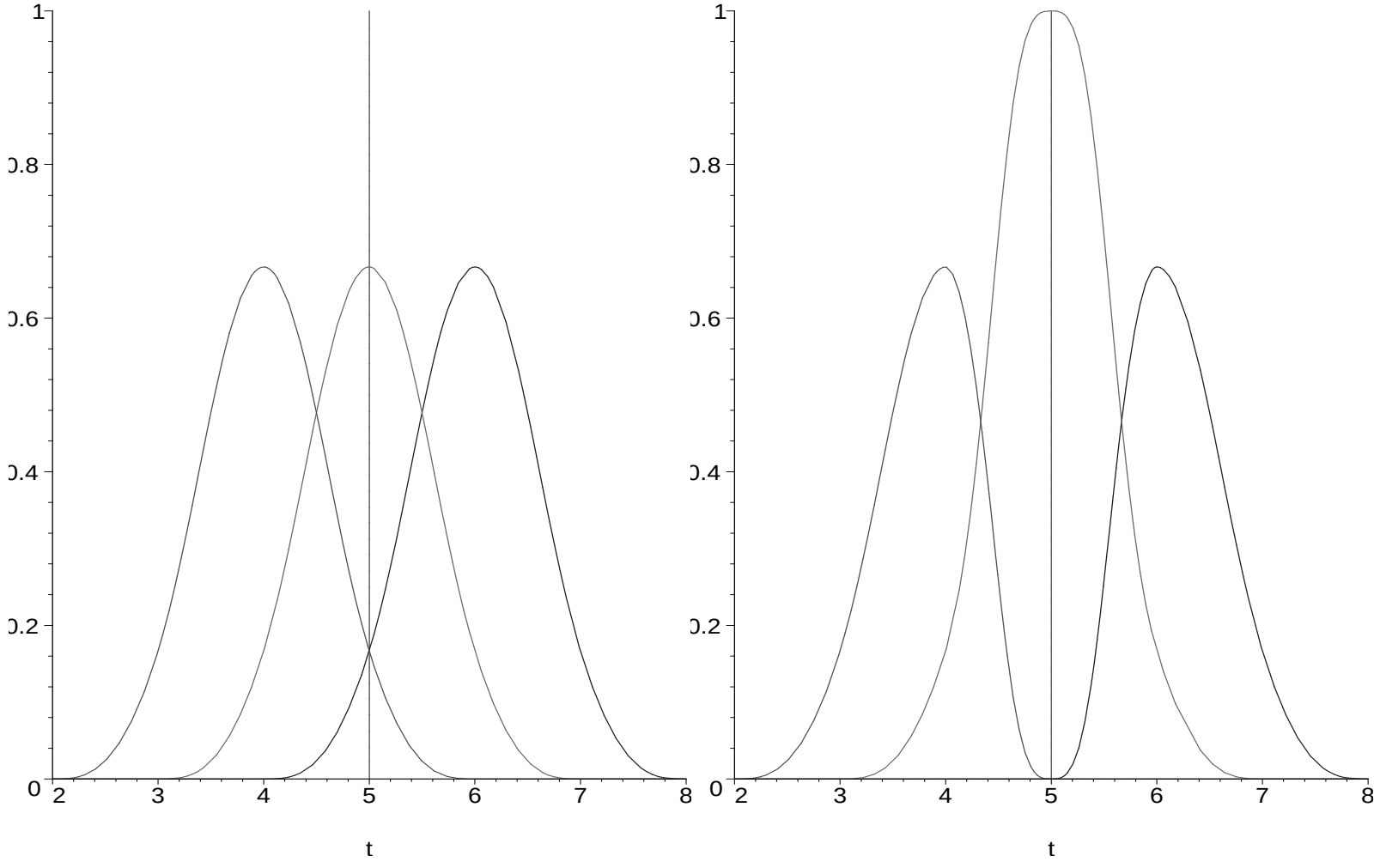
Will use them to modify the blending functions $F_k(t)$ like so.



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2. When all $s_k = 1$, we have all $p = 8$ and a basic approximating x-spline.
3. Define new zeroing points for the neighbours of $F_k(t)$:

$$T_{k-1}^{\text{Right}} := k + s_k$$
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$$T_{k-1}^{\text{Right}} := k + s_k$$
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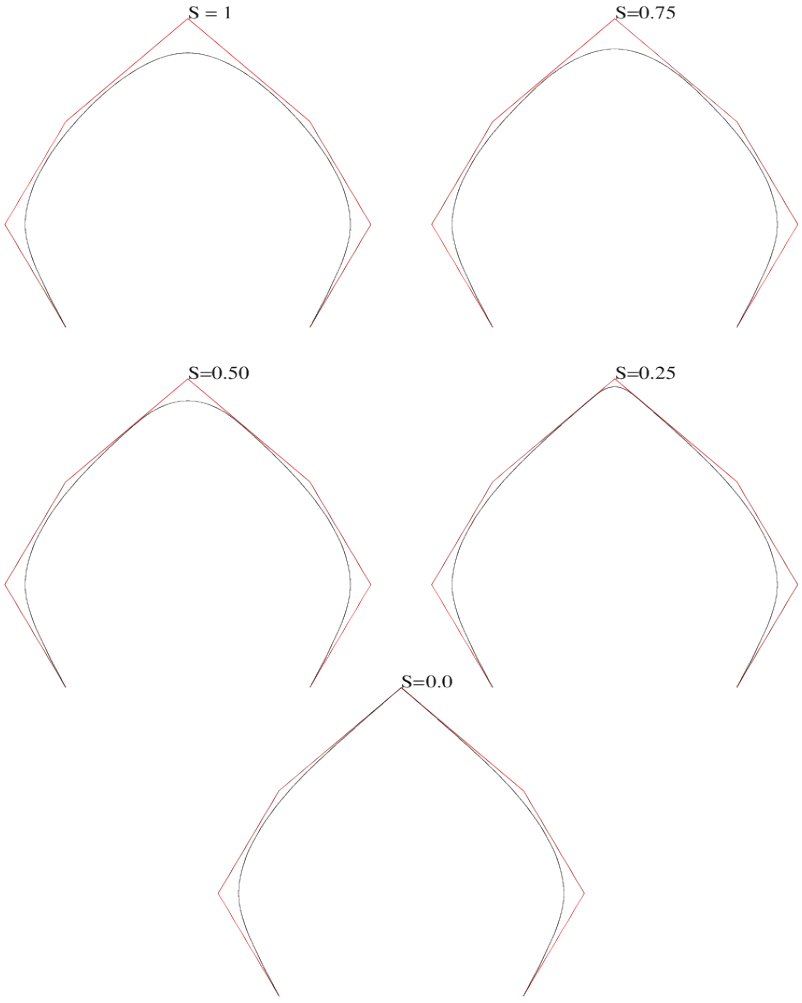
4. To preserve curvature (equality of second derivatives at $k-1$ and $k+1$), this requires setting special values of p

$$p_{k-1}^{\text{Right}} = 2(k - T_{k-1}^{\text{Right}})^2$$
$$p_{k+1}^{\text{Left}} = 2(k - T_{k+1}^{\text{Left}})^2$$

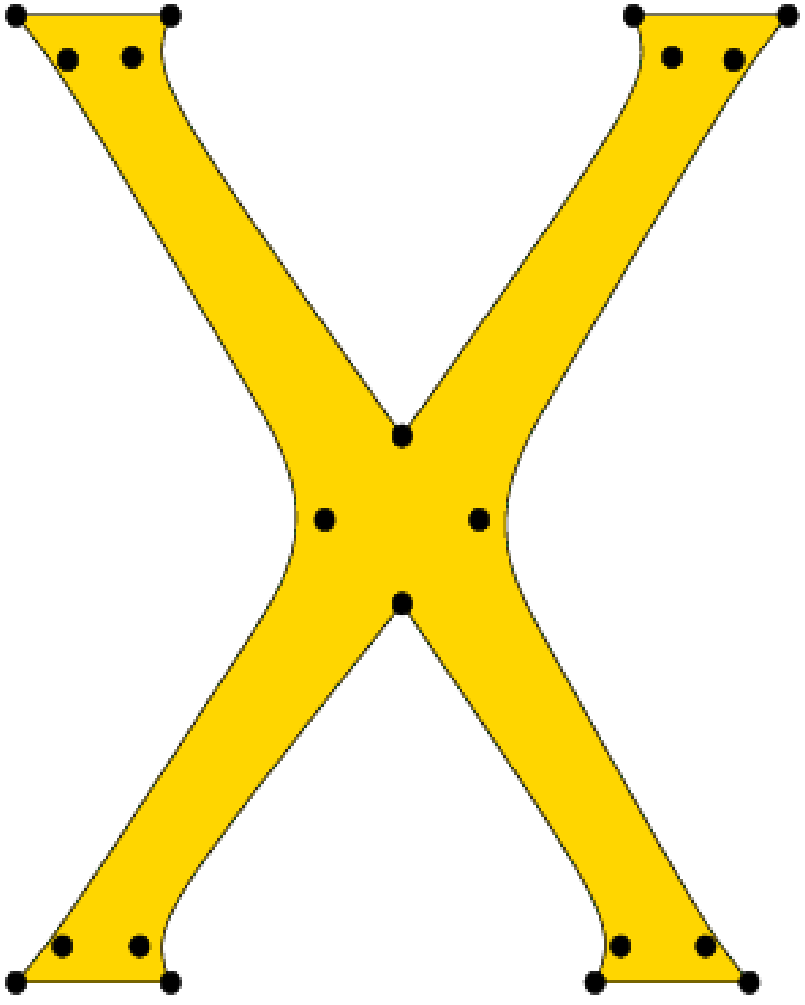
Can now interpolate sharply

End result: as s_k goes from 1 to 0 at a particular node, the spline gets closer to the node until it is interpolated sharply.



Fonts!

Good for drawing some basic fonts.



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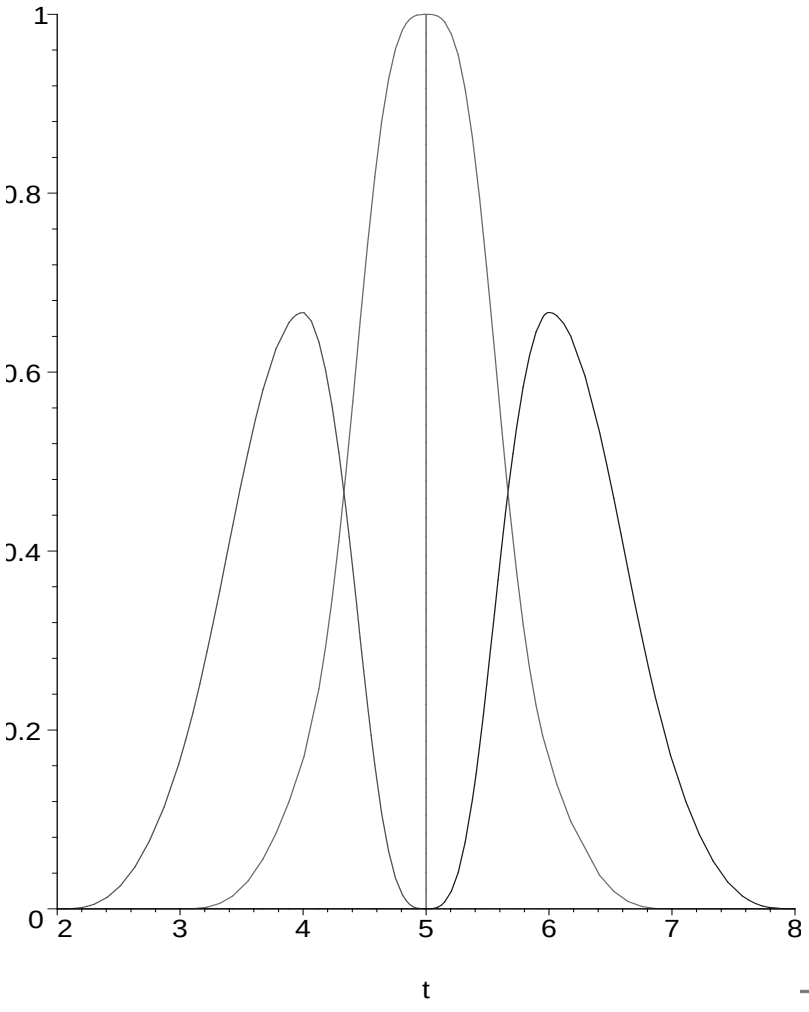
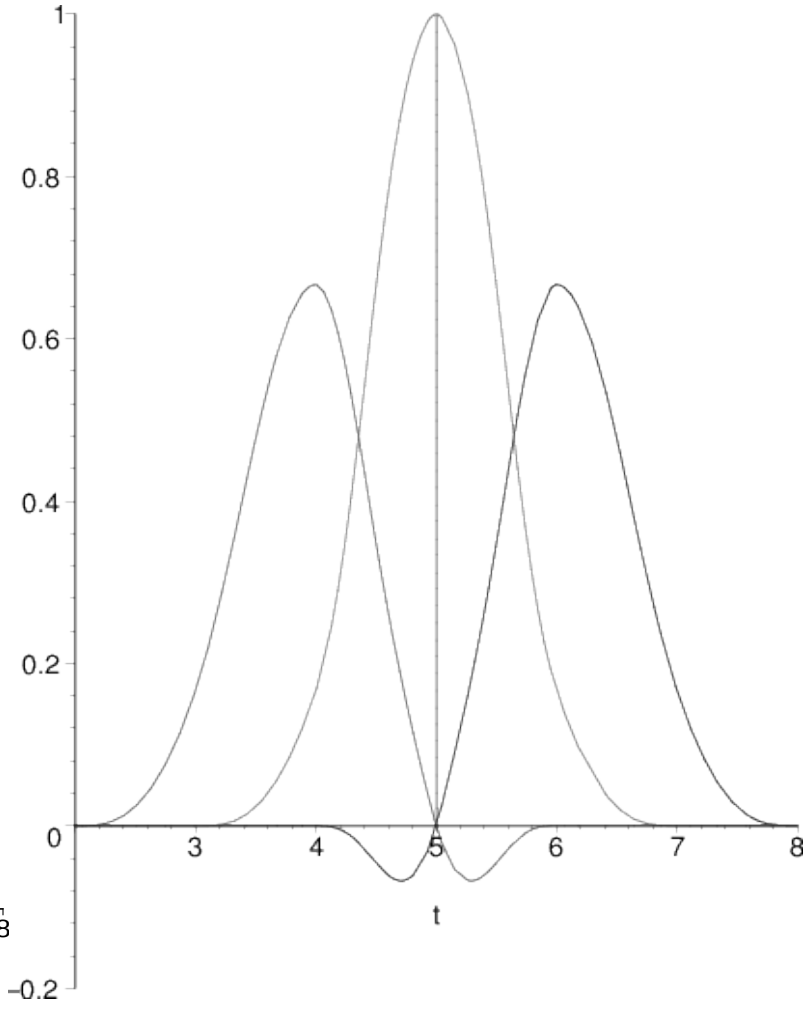
We'll keep the sum equal to 1, but we'll let the blending functions go negative now.
We've pushed the tails of the blending functions down to zero (via s_k parameters). We'll push those tails further down into the negative.

Pushing way down

Blending functions $F_k(t)$ will now look like this:





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This time use two polynomials, $g(u)$ for the positive part and $h(u)$ for the negative part.

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12 conditions on continuity, tangent line, and curvature:

$$
\begin{aligned}
h(-1) &= 0 & h'(-1) &= 0 & h''(-1) &= 0 \\
h(0) &= 0 & h'(0) &= q & h''(0) &= 4q \\
g(0) &= 0 & g'(0) &= q & g''(0) &= 4q \\
g(1) &= 1 & g'(1) &= 0 & g''(1) &= -2p
\end{aligned}
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Degree 5 solutions:

$$\begin{aligned} g(u) &= qu+2qu^2+(10-12q-p)u^3 \\ &\quad +(2p+14q-15)u^4+(6-5q-p)u^5 \\ h(u) &= qu+2qu^2-2qu^4+qu^5 \end{aligned}$$

Recovering Catmull-Rom

New q parameter to play with, corresponding to the s_k parameters:

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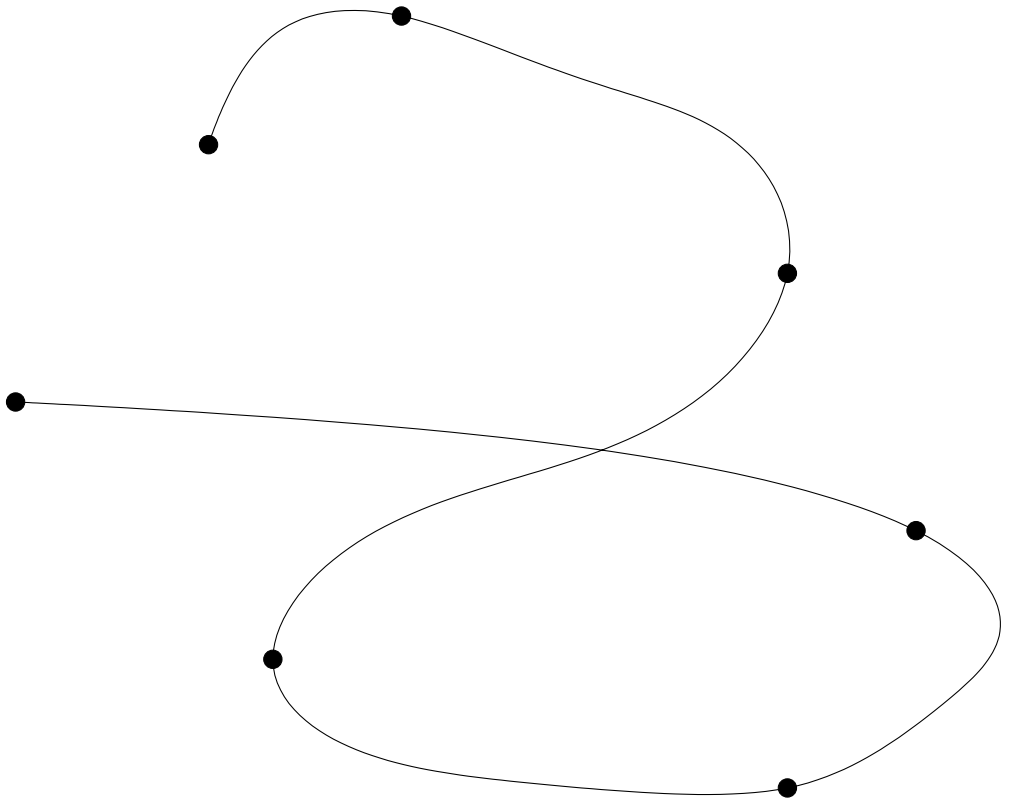
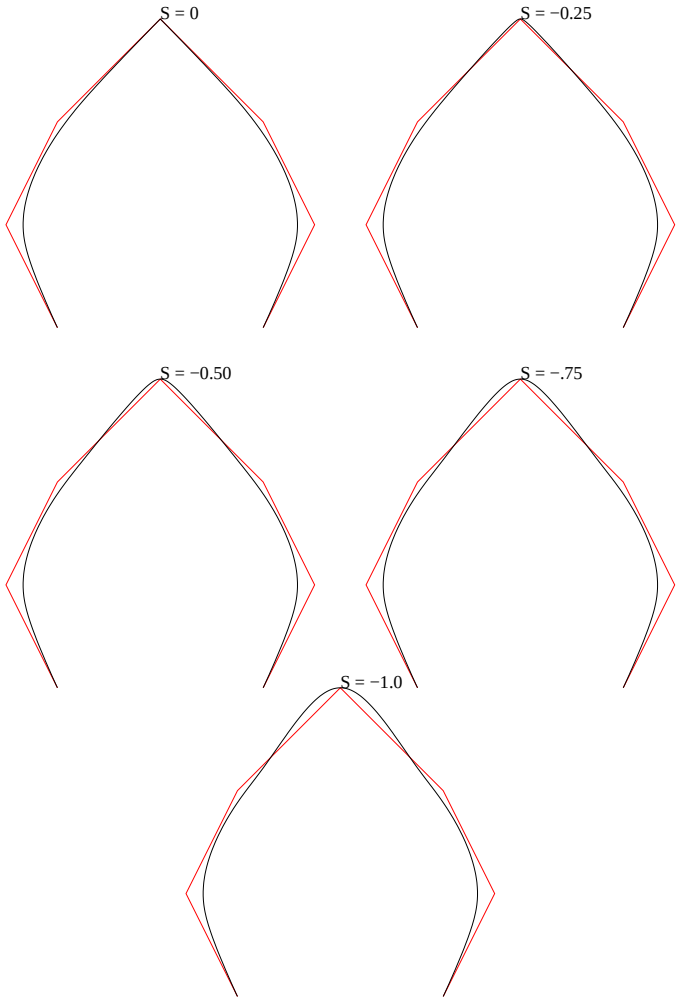
$$q_{k-1}^{\text{Right}} := -\frac{s_k}{2} =: q_{k+1}^{\text{Left}}$$

When $q = 1/2$ everywhere (i.e. $s = -1$ everywhere), we get an almost exact copy of Catmull-Rom.



Letting *s* vary slowly

Smoothly goes from sharp interpolation to smooth interpolation:



Thanks!

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